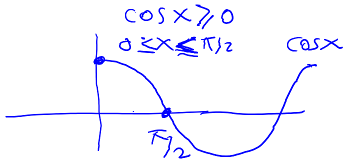
$(ab)^2 = a^2 b^2$

$\sin^2\theta + \cos^2\theta = 1$

$\sqrt{x^2} = |x|$

$\cos x \geq 0$, $0 \leq x \leq \pi/2$



Section 7.3: Trigonometric substitution

Objective: In this lesson, you learn

☐ How to evaluate integrals of certain forms using trigonometric substitutions.

I. Trigonometric Substitution.

$y = \sqrt{a^2 - x^2}$

$y^2 = a^2 - x^2$

$x^2 + y^2 = a^2$ A circle

center $= (0,0)$

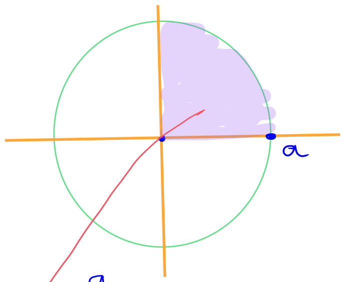
$r = a$

Problem: Find the area of a circle or an ellipse. i.e.

$$\int \sqrt{a^2 - x^2}\, dx \text{ where } a > 0?$$

If we change the variable from x to θ by the substitution

$\boxed{x = a \sin\theta} \to \boxed{dx = a\cos\theta\, d\theta}$



$a^2 - x^2 = a^2 - (a\sin\theta)^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta)$

so, $\boxed{a^2 - x^2 = a^2\cos^2\theta}$

$\int_0^a \sqrt{a^2-x^2}\, dx = \frac{\pi r^2}{4} = \frac{\pi a^2}{4}$

$\int \sqrt{a^2-x^2}\, dx = \int \sqrt{a^2\cos^2\theta} \cdot a\cos\theta\, d\theta$

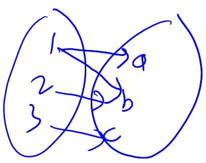
$= \int a\cos\theta \cdot a\cos\theta\, d\theta = a^2 \int \cos^2\theta\, d\theta$ when $0 \leq \theta \leq \pi/2$

Note that the difference between the substitution $u = a^2 - x^2$ and $x = a\sin\theta$ is

- $u = a^2 - x^2$, u (new) is a function of (old) x.
- $x = a\sin\theta$, x (old) is a function of (new) θ.

In general, to integrate $\int f(x)dx$, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse, called an **inverse substitution.** To make the calculations simpler, assume that g has an **inverse function**, that is, $x = g(t)$ is a one-to-one function. So if $x = g(t)$, then

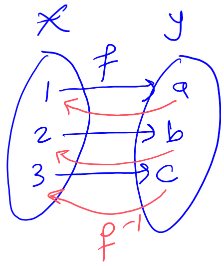
$$\int f(x)\ dx = \int f(g(t))\, g'(t)\, dt.$$



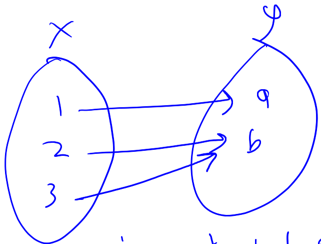
This is not a function

$(1,a)$, $(1,b)$

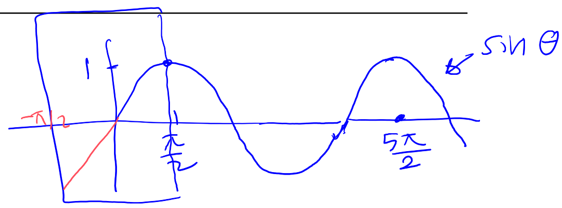
$(2,b)$, $(3,c)$



f is one-to-one



is not 1-1 function



Example 1: Integrate $\int \sqrt{a^2 - x^2}\, dx$

$\boxed{x = a\sin\theta} \rightarrow \boxed{dx = a\cos\theta \, d\theta}$

$\boxed{a^2 - x^2 = a^2\cos^2\theta}$

$\cos\theta \geq 0$

$0 \leq \theta \leq \frac{\pi}{2}$

$$\int \sqrt{a^2-x^2}\, dx = \int \sqrt{a^2\cos^2\theta} \cdot a\cos\theta \, d\theta$$

$$= \int a\cos\theta \cdot a \cdot \cos\theta \, d\theta$$

$$= a^2 \int \cos^2\theta \, d\theta$$

$\boxed{\cos^2 x = \frac{1}{2}(1+\cos 2x)}$

$$= a^2 \int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\int 1+\cos 2\theta \, d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$\boxed{\sin 2x = 2\sin x \cos x}$

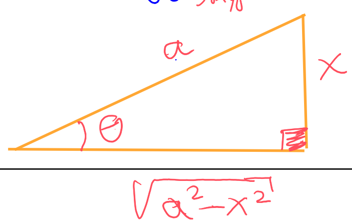
$$= \frac{a^2}{2}\left[\theta + \frac{1}{\cancel{2}} \cdot \cancel{2}\sin\theta\cos\theta\right] + C$$

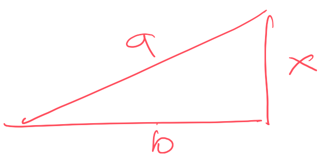
$$= \frac{a^2}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$\boxed{= \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\,\frac{\sqrt{a^2-x^2}}{a}\right] + C}$$

$x = a\sin\theta$

$\sin^{-1}\left(\frac{x}{a}\right) = \theta \longleftarrow \frac{x}{a} = \sin\theta$ (x → opp., a → hyp.)

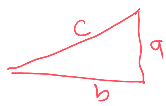




$b^2 + x^2 = a^2$

$b^2 = a^2 - x^2$

$b = \sqrt{a^2-x^2}$



$c^2 = a^2 + b^2$

Trigonometric substitution:

Here are trigonometric substitutions:

a. For the expression $\sqrt{a^2 - x^2}$, make a substitution $x = a\sin\theta$ defined on $[-\pi/2,\ \pi/2]$, and use the identity $1 - \sin^2\theta = \cos^2\theta$.

b. For the expression $\sqrt{a^2 + x^2}$, make a substitution $x = a\tan\theta$ defined on $(-\pi/2,\ \pi/2)$, and use the identity $1 + \tan^2\theta = \sec^2\theta$.

c. For the expression $\sqrt{x^2 - a^2}$, make a substitution $x = a\sec\theta$ defined on $[0,\ \pi/2)$ or $[\pi,\ 3\pi/2)$, and use the identity $\sec^2\theta - 1 = \tan^2\theta$.

Example 2: Evaluate $\int_0^1 x^3\sqrt{1 - x^2}\,dx$.

a^2

$x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \pi/2$

$\left.\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3}\right|_0^{\pi/2} = (0) - \left(\frac{1}{5} - \frac{1}{3}\right)$

$-\frac{\pi}{2} \le \theta \le \pi/2$

$\boxed{x = \sin\theta} \longrightarrow \boxed{dx = \cos\theta\, d\theta}$

$\boxed{1 - x^2 = 1 - \sin^2\theta = \cos^2\theta}$

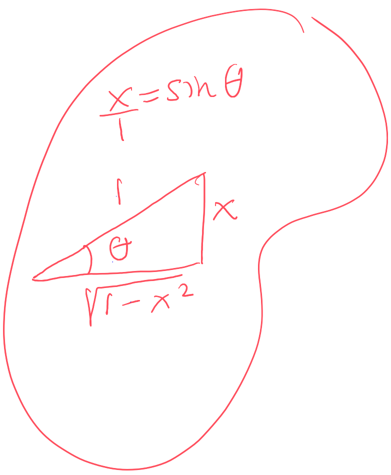
$$\int x^3\sqrt{1-x^2}\,dx = \int (\sin\theta)^3\sqrt{\cos^2\theta}\cdot\cos\theta\,d\theta$$

$$= \int \sin^3\theta\cdot\cos\theta\cos\theta\,d\theta$$

$$= \int \sin^3\theta\cos^2\theta\,d\theta$$

$$= \int \sin^2\theta\cos^2\theta\sin\theta\,d\theta$$

$$= \int (1 - \cos^2\theta)\cos^2\theta\sin\theta\,d\theta$$



$u = \cos\theta \longrightarrow du = -\sin\theta\,d\theta$

$$= -\int (1 - u^2)u^2\,du = -\int u^2 - u^4\,du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3} + C$$

$$= \frac{\left(\sqrt{1-x^2}\right)^5}{5} - \frac{\left(\sqrt{1-x^2}\right)^3}{3}$$

$$\int_0^1 x^3\sqrt{1-x^2}\,dx = \left.\frac{\left(\sqrt{1-x^2}\right)^5}{5} - \frac{\left(\sqrt{1-x^2}\right)^3}{3}\right|_0^1 = (0) - \left(\frac{1}{5} - \frac{1}{3}\right)$$

$$= \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

$$(\tan^{-1} x)' = \frac{1}{1+x^2}$$

Example 3: Evaluate $\int \frac{1}{1+x^2} dx$.

$$x = \tan\theta \longrightarrow dx = \sec^2\theta \, d\theta$$

$$1+x^2 = 1+\tan^2\theta = \sec^2\theta$$

$$x = \tan\theta$$
$$\theta = \tan^{-1}(x)$$

$$\int \frac{1}{1+x^2} dx = \int \frac{1}{\cancel{\sec^2\theta}} \cdot \cancel{\sec^2\theta} \, d\theta$$

$$= \int 1 \cdot d\theta = \theta$$
$$= \tan^{-1}(x) + C.$$

Example 4: Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx \longrightarrow a^2=4=2^2 \Rightarrow \boxed{a=2}$

$x=2\tan\theta \longrightarrow \boxed{dx=2\sec^2\theta\, d\theta}$

$\boxed{x^2=4\tan^2\theta}$

$\boxed{x^2+4}=(2\tan\theta)^2+4=4\tan^2\theta+4=4(1+\tan^2\theta)$
$=\boxed{4\sec^2\theta}$

so, $\sqrt{x^2+4}=\sqrt{4\sec^2\theta}=2\sec\theta$

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{4\tan^2\theta\cdot \cancel{2}\cancel{\sec\theta}}\cdot\cancel{2}\sec^{\cancel{2}}\theta\, d\theta$$

$$=\frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$=\frac{1}{4}\int \frac{\frac{1}{\cancel{\cos\theta}}}{\frac{\sin^2\theta}{\cancel{\cos^2\theta}}}\,d\theta$$

$$\frac{1}{\frac{a}{b}}=\frac{b}{a}$$

$\frac{\cos\theta}{\sin\theta}=\cot\theta$

$\frac{1}{\sin\theta}=\csc\theta$

$$=\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$1\div\frac{a}{b}=1*\frac{b}{a}$

$u=\sin\theta \qquad du=\cos\theta\, d\theta$

$$=\frac{1}{4}\int \frac{1}{u^2}\,du=\frac{1}{4}\cdot\frac{-1}{u}=\frac{-1}{4\sin\theta}=\frac{-1}{4}\csc\theta+C$$

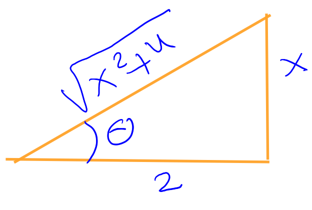
OR

$$=\frac{1}{4}\int \cot\theta\cdot\csc\theta\, d\theta=\frac{-1}{4}\csc\theta+C$$

$x=2\tan\theta$

$\frac{x}{2}=\tan\theta$

$$=\frac{-1}{4}\,\frac{\sqrt{x^2+4}}{x}+C$$



$ax^2+bx+c=0$

to complete a square

$\left(-\frac{b}{2}\right)^2 = \frac{b^2}{4}$

Example 5: Evaluate $\displaystyle\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx.$

$b=-2 \Rightarrow \left(\frac{-b}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1$

$x^2-2x+2 = (x^2-2x+1)+1 = (x-1)^2+1$

$$\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx = \int \frac{x^2+1}{((x-1)^2+1)^2}\,dx$$

$t = x-1 \Rightarrow dt = dx$

$(t+1)^2+1 = t^2+2t+2$

$$= \int \frac{(t+1)^2+1}{(t^2+1)^2}\,dt$$

let $t=\tan\theta \rightarrow \boxed{t^2+1 = \tan^2\theta+1 = \sec^2\theta}$

$\boxed{t^2=\tan^2\theta}$ $\boxed{2t = 2\tan\theta}$

$\boxed{dt = \sec^2\theta\, d\theta}$

$$\int \frac{(t+1)^2+1}{(t^2+1)^2}\,dt = \int \frac{t^2+2t+2}{(t^2+1)^2}\,dt = \int \frac{\tan^2\theta+2\tan\theta+2}{(\sec^2\theta)^{\cancel{2}}}\,\cancel{\sec^2\theta}\,d\theta$$

$\dfrac{\sin^2\theta}{\cancel{\cos^2\theta}} \Big/ \dfrac{1}{\cancel{\cos^2\theta}}$

$$= \int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta + 2\int \frac{\tan\theta}{\sec^2\theta}\,d\theta + \int \frac{2}{\sec^2\theta}\,d\theta$$

$\dfrac{\tan\theta}{\sec^2\theta} = \dfrac{\sin\theta}{\cancel{\cos\theta}} \Big/ \dfrac{1}{\cos^2\theta}$

$$= \int \sin^2\theta\,d\theta + 2\int \sin\theta\cos\theta\,d\theta + 2\int \cos^2\theta\,d\theta$$

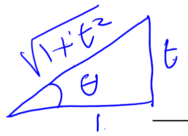
$$= \int \sin^2\theta\,d\theta + 2\int \sin\theta\cos\theta\,d\theta + \int\cos^2\theta\,d\theta + \int \cos^2\theta\,d\theta$$

$$= \int \underbrace{\sin^2\theta+\cos^2\theta}_{1}\,d\theta + 2\int \sin\theta\cos\theta\,d\theta + \int \cos^2\theta\,d\theta$$

$$= \theta + \sin^2\theta + \frac{1}{2}\int 1+\cos 2\theta\,d\theta$$

$$= \theta + \sin^2\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)$$

$\tan\theta = t$



$$= \boxed{\frac{3}{2}\theta + \sin^2\theta + \frac{1}{4}\sin\theta\cos\theta + C}$$

$$= \frac{3}{2}\tan^{-1}(t) + \frac{t^2}{1+t^2} + \frac{1}{4}\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}}$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \frac{3}{2}\tan^{-1}(x-1) + \frac{(x-1)^2}{1+(x-1)^2} + \frac{(x-1)}{4(1+(x-1)^2)}$$

Example 6: Evaluate $\int \frac{1}{x^2 (x^2-9)^{1/2}} dx.$

$\to a^2 = 3^2$

$x = 3\sec\theta \Rightarrow \boxed{dx = 3\sec\theta\tan\theta\, d\theta}$

$\boxed{x^2 = 9\sec^2\theta}$ $\qquad \boxed{x^2-9} = 9\sec^2\theta - 9 = 9(\sec^2\theta - 1) = \boxed{9\tan^2\theta}$

$$\int \frac{1}{x^2(x^2-9)^{1/2}}\, dx = \int \frac{1}{\cancel{9}\,3\sec^2\theta \cdot (9\tan^2\theta)^{1/2}}\, \cancel{3\sec\theta}\tan\theta\, d\theta$$

$$= \int \frac{1}{3\sec\theta \cdot 3\cancel{\tan\theta}}\, \cancel{\tan\theta}\, d\theta$$

$$= \frac{1}{9}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{9}\int \cos\theta\, d\theta$$

$$= \frac{1}{9}\sin\theta + C$$

$$= \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

$x = 3\sec\theta$

$\sec\theta = \frac{x}{3}$